Announcements

1) Candidate talk 3-4

   CB 2070

2) Skipping around a bit in book

   Today 23, 1^{st} part of 24,

   then 25 (series later)

Theorem: (monotone convergence)

If $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence and $a_n \leq a_{n+1}$ for all $n\in\mathbb{N}$ (a_n is monotone increasing), then $(a_n)_{n\in\mathbb{N}}$ converges to its least upper bound proof: Let L be the least upper bound of the set

$$S = \{a_n \mid n \in \mathbb{N}\}.$$

Choose $\varepsilon > 0$. By definition of the least upper bound, there is an $x \in S$ with

$L < x + \varepsilon$ we can rewrite as $L - x < \varepsilon$.

Since $x \in S$, $\exists$ $N \in \mathbb{N}$ with $x = a_N$. So rewriting again,

$$L - a_N < \varepsilon.$$

Now let $n \in \mathbb{N}$, $n \geq N$.

Since the sequence $(a_n)_{n \in \mathbb{N}}$ is monotone increasing,

$$a_n \geq a_N \quad \forall \; n \geq N.$$

$$L - a_n$$

$$= L - a_n + a_N - a_N$$

$$= L - a_N + (a_N - a_n)$$

But $a_n \geq a_N$, so $(a_N - a_n) \leq 0$.

So

$$L - a_n \leq L - a_N + 0 = L - a_N$$
$$< \varepsilon$$

Since $L = \sup(S)$, $L \geq a_n \ \forall n \in \mathbb{N}$,

so $\varepsilon > L - a_n = |L - a_n|$ $\square$

Corollary: If $(a_n)_{n\in\mathbb{N}}$ is bounded and $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$ (the sequence is monotone decreasing), then $(a_n)_{n\in\mathbb{N}}$ converges to its greatest lower bound

proof: Let $b_n = -a_n$, a monotone increasing sequence. Apply previous theorem to $(b_n)_{n\in\mathbb{N}}$ and use HW problem $\sup(-S) = -\inf(S)$ $\square$

Example 1: (recursive sequences)

Let $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$.

Prove that 1) $a_n < 2 \quad \forall n \in \mathbb{N}$

2) $(a_n)_{n \in \mathbb{N}}$ increasing.

Then the limit will exist by monotone convergence - find the limit!

1) by induction.

$1 < 2$, so holds for a_1

Now assume $a_n < 2$

Then $a_{n+1} = \sqrt{2+a_n}$

$$< \sqrt{2+2}$$

$$= \sqrt{4} = 2.$$

Done by induction.

2) $(a_n)_{n \in \mathbb{N}}$ is monotone increasing

$a_1 = 1$, $a_2 = \sqrt{1+2} = \sqrt{3} > 1$,

so holds for $n=1$.

Now assume $a_n \geq a_{n-1}$

Then

$$a_{n+1} = \sqrt{2+a_n} \geq \sqrt{2+a_{n-1}}$$

$$= a_n ,$$

again done by induction

So, since $a_n \geq 1$ for all $n \in \mathbb{N}$,

by the monotone convergence theorem, $(a_n)_{n \in \mathbb{N}}$ converges to its least upper bound L.

What is L? We know

$$a_{n+1} = \sqrt{2 + a_n}$$

Square both sides.

$$(a_{n+1})^2 = 2 + a_n$$

$$(a_{n+1})^2 = 2 + a_n$$

$a_n \to L$ as $n \to \infty$, so

$a_{n+1} \to L$ as well.

It is true (but we won't prove it today) that $(a_{n+1})^2 \to L^2$ as $n \to \infty$. Taking limits on both sides,

$L^2 = 2 + L$. Then

$L^2 - L - 2 = 0$, using quadratic fomula,

$$L = \frac{1 \pm \sqrt{9}}{2}$$

$$(L^2 - L - 2 = 0)$$

$L = \dfrac{1 \pm 3}{2}$, but can't be negative, so

$L = 2$.

Properties of Sequential Limits

Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$

and $c \in \mathbb{R}$. Then

1) $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$

2) $\lim_{n\to\infty} (c a_n) = cL$

3) $\lim_{n\to\infty} (a_n b_n) = L \cdot M$

(used in previous example)

4) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ provided

$M \neq 0$

5) If $a_n \leq b_n$ for all

$n \geq N \in \mathbb{N}$, then

$L \leq M$. ($N=1$, this

means $a_n \leq b_n$ $\forall$ $n \in \mathbb{N}$.)

All of these properties are yours to use!

Squeeze Theorem: Suppose

$a_n \le b_n \le c_n$ and

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$.

Then $\lim_{n \to \infty} b_n = L$

Proof tomorrow